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Identification methods in nonlinear heat conduction. Part I: model reduction

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Abstract

An original method for the identification of Reduced Models (RM) of nonlinear diffusive thermal systems is proposed in this numerical study. This method derives from the Modal Identification Method developed for linear systems in previous works. Starting from a Detailed Model (DM) under matrix form, a RM structure is defined. RM's matrices are then identified through the minimization of a squared residues functional built with the discrepancy between system responses (DM outputs in this study) on the one hand and RM outputs on the other hand, when a specific input signal is applied. A tridimensional example with thermal conductivity depending linearly on temperature illustrates the method. In comparison with DM, computing time is drastically reduced (division by a factor greater than 1000 in the proposed example) without significant loss of accuracy.

1. Introduction

Both natural and industrial thermal systems frequently exhibit nonlinear behaviour, because of thermo-dependant physical properties, presence of radiation or velocity–temperature coupling in the case of natural convection. When tridimensional aspects cannot be ignored, classical numerical methods based on a spatial discretization of the studied domain (finite elements, finite volumes, finite differences, etc.), induce large systems of nonlinear equations whose resolution requires huge computing time. Although computers are more and more powerful, studied systems become more complex and computing time reduction remains a crucial issue, especially for control command processes for example.

Model reduction methods provide interesting answers, as they allow to replace a large size model, also called Detailed Model (DM), whose order N corresponds to the number of discretization nodes, by a small size model or Reduced Model (RM) whose order is $n \ll N$. RMs are made up of small systems of equations; they simulate the system thermal behaviour for the whole domain or a part of it with limited loss of accuracy and very short computing time.

While numerous reduction methods exist for linear systems, including modal methods [1–4], internal balanced representation method [5], Eitelberg method [4,6], nodal topology reduction method [7], modal base reconstruction methods as the modal amalgam method [8] and the Modal Identification Method [9], the number

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Nomenclature

Latin characters

A, B, C, F, G, H, M, T, W, X, Y, Z	matrices and vectors
C_p	specific heat, $\mathrm{J\,kg^{-1}\,K^{-1}}$
J_{red}	criterion to minimise for the RM identification
DM, RM	Detailed Model, Reduced Model
dimz(k)	$= k(k+1)/2$, k integer
N	DM order i.e. number of discretization nodes
n	RM order
nt	number of time steps for temperature data used for RM identification
p_v	volumetric heat source distribution, $\mathrm{W\,m^{-3}}$
q	output vector dimension
t	time, s
U	thermal input

Greek symbols

λ	thermal conductivity, $\mathrm{W\,m^{-1}\,K^{-1}}$
ρ	density, $\mathrm{kg\,m^{-3}}$
Ω, Ψ	matrix and vector

Superscript

$\bullet$	derivation with respect to time

of reduction methods for nonlinear systems is limited. Among these techniques, one should mention the Karhunen–Loève–Galerkin method [10], based on statistical considerations and also known as proper orthogonal decomposition (POD). This method has been applied to conduction problems [11] as well as to natural convection problems [12]. In the "branch modes method", the model reduction is performed in a branch base solution of a branch problem defined from thermophysical properties mean values. This technique has been applied to a heating wire [13]. Another method consists in operating the reduction in the Lanczos vectors base [14]. In these three methods, governing equations have to be entirely known, including thermophysical parameters.

In the case of nonlinear diffusive thermal systems, a new reduction method is proposed in this paper. It is in fact a low order model identification method. Our approach requires the knowledge of the equations structure and temperature evolutions in locations of interest, constituting responses to a specific input signal. If these temperature are obtained by measurements, the method can be used for experimental modelling and values of equations parameters may be unknown. However, in the present study, temperature data are generated through a DM (governing equations as well as their parameters are known in that case).

2. Methodology summary

The proposed method derives from the Modal Identification Method [9,15] developed for linear systems. Starting from DM structure, RM structure is defined. Elements (matrices) of this RM structure have then to be identified. The identification principle is based on the minimization of a squared residues functional built with the discrepancy between system responses (DM outputs in this study) on the one hand and RM outputs on the other hand, when a specific input signal is applied. One should note the existence of reduction methods for nonlinear systems using such an optimization approach [16]. In fact RM acts as a transfer function linking a single thermal input to considered outputs. Up to now, the method is therefore limited to mono-input/multi-outputs systems. Thanks to the adopted state space representation, it is independent of geometry and can be applied to multidimensional systems. Reduced Models work in both transient and steady regimes.

3. Justification of RM structure

3.1. Hypothesis

Let us assume a system composed of a purely diffusive medium whose thermal conductivity varies with temperature. Density and specific heat are supposed to be invariant with temperature. Transient energy equation governing heat transfer through the domain is written:

$$\rho(r)C_p(r)\frac{\partial T}{\partial t}(r,t) = \mathrm{div}(\lambda(r,T)\,\vec{\mathrm{grad}}\,T(r,t)) + p_v(r,t) \tag{1}$$

After a spatial discretization, Eq. (1) and associated boundary conditions can be written under a discrete matrix form:

$$\dot{T}(t) = A(T)T(t) + BU(t) \tag{2}$$

where $T(t)$ (dim. N) is the vector containing temperatures of all discretization nodes, $\dot{T}(t)$ is the derivative of vector T with respect to time, matrix A (dim. N, N) is the state matrix depending on temperature because of the thermal conductivity, $U(t)$ is a thermal input (a

heat flux density or an internal heat source for instance) and vector B (dim. N) is the command or input vector linking nodes to $U(t)$.

Note: although Eq. (2) can be written for a vector U including several thermal inputs, the presented reduction method is limited to only one time-varying excitation. The others are assumed to be constant (here equal to zero).

A matrix C (dim. q, N) with $q \leqslant N$, called observation or output matrix, can be used to select a part of the whole temperature field T. This selection is contained in output vector $Y(t)$ (dim. q), then written:

$$Y(t) = CT(t) \tag{3}$$

Let us assume that thermal conductivity can be written as the sum of two terms:

$$\lambda(r, T) = \lambda_0(r) + f(r, T) \tag{4}$$

where f is a function of position r and local temperature T and $\lambda_0(r)$ is a conductivity reference distribution.

By separating linear and nonlinear terms in Eq. (2) according to Eq. (4) and by using Eq. (3), one can write a DM known as "*State Space Representation*", linking $Y(t)$ to $U(t)$:

$$\begin{cases} \dot{T}(t) = AT(t) + BU(t) + \Psi(T(t)) \\ Y(t) = CT(t) \end{cases} \tag{5a,b}$$

where matrix A (dim. N, N) is now the state matrix of a linear system with same geometry but constant thermal conductivity field $\lambda_0(r)$, $\Psi(T(t))$ (dim. N) is the vector gathering nonlinearities, hence corresponding to term $f(r, T)$ of Eq. (1). For each node i, component Ψ_i of vector Ψ is the sum of nonlinear contributions of node i with its neighbouring nodes.

It should be noted that a DM can always be written under the form of Eqs. (5a,b), whatever the discretization method (finite elements, finite volumes, finite differences, ...).

In the following, it is assumed that thermal conductivity is linearly variable with temperature, i.e. Eq. (4) can be written:

$$\lambda(r, T) = \lambda_0(r) + \beta(r)T \tag{6}$$

Remark: Although the method is developed for linear variation of thermal conductivity with temperature, one should note that analogous developments could be written in the more general hypothesis of a polynomial law, as it will be shown in Section 3.2.

3.2. Formulation of RM equations

Let us imagine that we solve the eigenvalue problem associated with matrix A of Eq. (5a). Let us call F the diagonal matrix whose components are the N eigenvalues of A, and M (dim. N, N) the matrix whose columns are eigenvectors of A.

Transformation $T(t) = MX(t)$ injected in Eq. (5a,b) leads to:

$$\begin{cases} \dot{X}(t) = FX(t) + GU(t) + M^{-1}\Psi(MX(t)) \\ Y(t) = HX(t) \end{cases} \tag{7a,b}$$

with:

$$\begin{cases} F = M^{-1}AM \\ G = M^{-1}B \\ H = CM \end{cases} \tag{8a,b,c}$$

Components of new state vector $X(t)$ are states $X_i(t)$, $i \in \langle 1, N \rangle$, which are coupled through vector of nonlinearities Ψ. It will now be shown how to rewrite vector $\Psi(MX(t))$ as:

$$\Psi(T(t)) = \Psi(MX(t)) = LZ(X(t)) \tag{9}$$

where vector $Z(X(t))$ depends only on couplings between states $X_i(t)$ and contains, according to thermal conductivity law (6), products $X_i(t)X_j(t)$:

$$\begin{aligned} Z(X(t)) = \big[& X_1^2(t)\ X_1(t)X_2(t)\ X_1(t)X_3(t)\ \cdots\ X_1(t)X_N(t) \\ & X_2^2(t)\ X_2(t)X_3(t)\ \cdots\ X_2(t)X_N(t)\ \cdots \\ & \cdots\ X_{N-1}^2(t)\ X_{N-1}(t)X_N(t)\ X_N^2(t) \big]^{\mathrm{T}} \end{aligned} \tag{10}$$

Dimension of vector $Z(X(t))$ is therefore $\mathrm{dimz}(N) = N(N+1)/2$. Dimension of matrix L is $(N, \mathrm{dimz}(N))$.

To illustrate following developments with brevity and clarity, a 3D cartesian problem is considered, with ρ and C_p independent of position r, as well as λ_0 and β in Eq. (6). Nevertheless no generality is lost concerning other systems of coordinates and cases involving $\rho(r)$, $C_p(r)$, $\lambda_0(r)$ and $\beta(r)$.

In that case, the transient energy equation is written:

$$\rho C_p \frac{\partial T}{\partial t}(r, t) = \mathrm{div}(\lambda(T)\, \vec{\mathrm{grad}}\ T(r, t)) \tag{11}$$

The term $\mathrm{div}(\lambda(T)\, \vec{\mathrm{grad}}\ T(r, t))$ of Eq. (11) can be written:

$$\begin{aligned} \mathrm{div}(\lambda(T)\, \vec{\mathrm{grad}}\ T(r, t)) = & \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda(T)\frac{\partial T}{\partial y}\right) \\ & + \frac{\partial}{\partial z}\left(\lambda(T)\frac{\partial T}{\partial z}\right) \end{aligned} \tag{12}$$

The following developments are based on the Finite Volumes Method, again without loss of generality concerning other discretization methods. Let us consider a control volume around a point **P**, using the following notations: **W**, **E**, **N**, **S**, **O** and **B** indicate points located respectively to the West, East, North, South, top (tOp) and bottom (Bottom) of point P and ***w***, ***e***, ***n***, ***s***, ***o*** and ***b*** indicate interfaces between the considered control

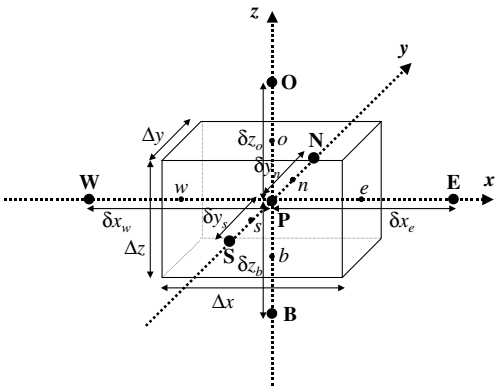

Fig. 1. Control volume.

volume and neighbouring control volumes. Fig. 1 illustrates this configuration.

Integration with respect to space, of the first term of the right part of Eq. (12), gives:

$$\int_{z=b}^{z=o}\int_{y=s}^{y=n}\int_{x=w}^{x=e}\left(\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{z=b}^{z=o}\int_{y=s}^{y=n}\left(\left(\lambda(T)\frac{\partial T}{\partial x}\right)_{x=e}-\left(\lambda(T)\frac{\partial T}{\partial x}\right)_{x=w}\right)\mathrm{d}y\,\mathrm{d}z \tag{13}$$

Let us introduce in Eq. (13) the thermal conductivity law defined by Eq. (6):

$$\begin{aligned}\int_{z=b}^{z=o}\int_{y=s}^{y=n}\int_{x=w}^{x=e}\left(\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \lambda_0\int_{z=b}^{z=o}\int_{y=s}^{y=n}\left(\left(\frac{\partial T}{\partial x}\right)_{x=e}-\left(\frac{\partial T}{\partial x}\right)_{x=w}\right)\mathrm{d}y\,\mathrm{d}z\\ &+\beta\int_{z=b}^{z=o}\int_{y=s}^{y=n}\left(\left(T\frac{\partial T}{\partial x}\right)_{x=e}-\left(T\frac{\partial T}{\partial x}\right)_{x=w}\right)\mathrm{d}y\,\mathrm{d}z\end{aligned} \tag{14}$$

which can also be written:

$$\begin{aligned}\int_{z=b}^{z=o}\int_{y=s}^{y=n}\int_{x=w}^{x=e}\left(\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \lambda_0\int_{z=b}^{z=o}\int_{y=s}^{y=n}\left(\left(\frac{\partial T}{\partial x}\right)_{x=e}-\left(\frac{\partial T}{\partial x}\right)_{x=w}\right)\mathrm{d}y\,\mathrm{d}z\\ &+\beta\int_{z=b}^{z=o}\int_{y=s}^{y=n}\left(\left(\frac{1}{2}\frac{\partial(T^2)}{\partial x}\right)_{x=e}\right.\\ &\left.-\left(\frac{1}{2}\frac{\partial(T^2)}{\partial x}\right)_{x=w}\right)\mathrm{d}y\,\mathrm{d}z\end{aligned} \tag{15}$$

In a classic way, it is assumed that quantities (heat flux densities) $\left(\lambda_0\frac{\partial T}{\partial x}\right)_{x=e}$ and $\left(\frac{\beta}{2}\frac{\partial(T^2)}{\partial x}\right)_{x=e}$ prevail on the entire interface e whose area is $\Delta y\Delta z$ [17].

As well, it is assumed that quantities (heat flux densities) $\left(\lambda_0\frac{\partial T}{\partial x}\right)_{x=w}$ and $\left(\frac{\beta}{2}\frac{\partial(T^2)}{\partial x}\right)_{x=w}$ prevail on the entire interface w whose area is $\Delta y\Delta z$.

The approximation of heat flux densities is made as follows:

$$\left(\lambda_0\frac{\partial T}{\partial x}\right)_{x=e}=\lambda_0\left(\frac{\partial T}{\partial x}\right)_{x=e}\approx\lambda_0\left(\frac{T_\mathrm{E}-T_\mathrm{P}}{\delta x_e}\right) \tag{16}$$

$$\left(\frac{\beta}{2}\frac{\partial(T^2)}{\partial x}\right)_{x=e}=\frac{\beta}{2}\left(\frac{\partial(T^2)}{\partial x}\right)_{x=e}\approx\frac{\beta}{2}\left(\frac{T_\mathrm{E}^2-T_\mathrm{P}^2}{\delta x_e}\right) \tag{17}$$

$$\left(\lambda_0\frac{\partial T}{\partial x}\right)_{x=w}=\lambda_0\left(\frac{\partial T}{\partial x}\right)_{x=w}\approx\lambda_0\left(\frac{T_\mathrm{P}-T_\mathrm{W}}{\delta x_w}\right) \tag{18}$$

$$\left(\frac{\beta}{2}\frac{\partial(T^2)}{\partial x}\right)_{x=w}=\frac{\beta}{2}\left(\frac{\partial(T^2)}{\partial x}\right)_{x=w}\approx\frac{\beta}{2}\left(\frac{T_\mathrm{P}^2-T_\mathrm{W}^2}{\delta x_w}\right) \tag{19}$$

By injecting Eqs. (16)–(19) into Eq. (15), we get:

$$\begin{aligned}\int_{z=b}^{z=0}\int_{y=s}^{y=n}\int_{x=w}^{x=e}\left(\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &\approx\lambda_0\left[\left(\frac{T_\mathrm{E}-T_\mathrm{P}}{\delta x_e}\right)-\left(\frac{T_\mathrm{P}-T_\mathrm{W}}{\delta x_w}\right)\right]\Delta y\Delta z\\ &+\frac{1}{2}\beta\left[\left(\frac{T_\mathrm{E}^2-T_\mathrm{P}^2}{\delta x_e}\right)-\left(\frac{T_\mathrm{P}^2-T_\mathrm{W}^2}{\delta x_w}\right)\right]\Delta y\Delta z\end{aligned} \tag{20}$$

This operation is realised for each internal node in the three directions. Expressions in the first term of Eq. (20) enter in the composition of matrix A of Eq. (5a). Expressions in the second term of Eq. (20) enter in the composition of vector Ψ. It can be seen that squares of temperatures at discretization nodes appear in nonlinear terms components of vector $\Psi(T)$. A similar treatment for boundary conditions allows to complete matrix A and vector Ψ, and to create matrix B. Diagonalization of matrix A gives matrices F and M.

Transformation $T(t) = MX(t)$ allows to write, for each square of temperature T_k at node k:

$$\begin{aligned}T_k^2&=\left(\sum_{i=1}^{N}M_{ki}X_i\right)\left(\sum_{j=1}^{N}M_{kj}X_j\right)\\&=\sum_{i=1}^{N}\sum_{j=1}^{N}(M_{ki}M_{kj}X_iX_j)\end{aligned} \tag{21}$$

Vector Ψ is therefore written as Eq. (9) with the definition of vector Z (Eq. (10)). Products $M_{ki}M_{kj}$ enter in the composition of matrix L.

The new model of the system is now:

$$\begin{cases}\dot{X}(t)=FX(t)+GU(t)+\Omega Z(X(t))\\ Y(t)=HX(t)\end{cases} \tag{22a,b}$$

where matrices F, G, H are defined by relations ((8a)–(c)) and matrix Ω $(N, \mathrm{dimz}(N))$ is $\Omega = M^{-1}L$.

At this time, it is important to note the following remarks:

1. Eq. (22a,b) constitute a DM in modal form, with the nonlinear term $\Omega Z(X(t))$.
2. A modal formulation of DM is interesting in the case of linear systems ($\Omega = 0$) because it allows a complete decoupling of states X_i. On the contrary, such a modal form is very penalising for nonlinear cases, because matrix Ω is $(N, \mathrm{dimz}(N))$ with $\mathrm{dimz}(N) = N(N+1)/2$ and can be therefore very large if N is important. For example, if $N = 1000$, Ω is (1000, 500 500). The formulation (22) becomes interesting when associated with a reduced state vector X whose dimension is $n \ll N$.
3. In a way similar to the Modal Identification Method developed for linear systems [9,15], we propose in the following a Reduced Model formulation based on Eq. (22a,b) with reduced matrices F, G, Ω and H to be identified. Therefore the eigenvalue problem associated with matrix A has not to be solved: matrices F and M are not computed, as well as matrices L and Ω.
4. A formulation analogous to Eq. (22a,b) can be obtained in the more general case of thermal conductivity depending on temperature according to a polynomial law. Components of vector Ψ are not only composed of squares of temperatures at discretization nodes, but also of all powers of temperatures up to $d+1$ with d the greatest power of the polynomial function. By applying $T = MX$, one makes appear X_iX_j products but also $X_iX_jX_k$ and $X_iX_jX_kX_l$ ones, etc. up to "$(d+1)$uples" products. For a given order N, the new vector $Z(X)$ is larger, as well as associated matrix Ω.
5. Moreover it seems possible to extend the method to the case of thermal capacity varying with temperature. A nonlinear unsteady term appears that can be included in vector Ψ, which is written $\Psi(T, \dot{T})$. For each node **P** of the spatial discretization, the unsteady term involves products of powers of temperature T_P with time derivative $\dot{T}_\mathrm{P}$. For example, if thermal capacity depends linearly on temperature, the term $T_\mathrm{P}\dot{T}_\mathrm{P}$ appears. Transformation $T = MX$ leads to:

$$T_\mathrm{P}\dot{T}_\mathrm{P} = \left(\sum_{i=1}^{N} M_{ki}X_i\right)\left(\sum_{j=1}^{N} M_{kj}\dot{X}_j\right) = \sum_{i=1}^{N}\sum_{j=1}^{N}(M_{ki}M_{kj}X_i\dot{X}_j) \tag{23}$$

A formulation similar to Eq. (22a,b) can then be obtained, with vector Z becoming $Z(X, \dot{X})$ in which are included the N^2 products $X_i\dot{X}_j$.

Let us now consider a partition of states X_i in Eq. (22a,b): a part X_D containing n ($<N$) dominant states and a part X_ND gathering the $N-n$ nondominant states:

$$\begin{cases} \begin{bmatrix} \dot{X}_\mathrm{D}(t) \\ \dot{X}_\mathrm{ND}(t) \end{bmatrix} = \begin{bmatrix} F_\mathrm{D} & 0 \\ 0 & F_\mathrm{ND} \end{bmatrix}\begin{bmatrix} X_\mathrm{D}(st) \\ X_\mathrm{ND}(t) \end{bmatrix} + \begin{bmatrix} G_\mathrm{D} \\ G_\mathrm{ND} \end{bmatrix}U(t) \\ \qquad\qquad + \begin{bmatrix} \Omega_\mathrm{D} \\ \Omega_\mathrm{ND} \end{bmatrix} Z\left(\begin{bmatrix} X_\mathrm{D}(t) \\ X_\mathrm{ND}(t) \end{bmatrix}\right) \\ Y(t) = \begin{bmatrix} H_\mathrm{D} & H_\mathrm{ND} \end{bmatrix}\begin{bmatrix} X_\mathrm{D}(t) \\ X_\mathrm{ND}(t) \end{bmatrix} \end{cases} \tag{24a,b}$$

Keeping only the n dominant states gives:

$$\begin{cases} \dot{X}_\mathrm{D}(t) = F_\mathrm{D}X_\mathrm{D}(t) + G_\mathrm{D}U(t) + \Omega_\mathrm{D}Z_\mathrm{D}(X_\mathrm{D}(t)) \\ \widehat{Y}(t) = H_\mathrm{D}X_\mathrm{D}(t) \end{cases} \tag{25a,b}$$

where vector $\widehat{Y}(t)$ approaches vector $Y(t)$ (the one computed using Eq. (5a,b) or (22a,b)), $X_\mathrm{D}(t)$ is a reduced state vector whose dimension is n (we wish $n \ll N$). $F_\mathrm{D}(n,n)$ is diagonal. $\Omega_\mathrm{D}(n, \mathrm{dimz}(n))$, $H_\mathrm{D}(q,n)$ and $G_\mathrm{D}(n)$ are the associated reduced matrices and vector. The vector $Z_\mathrm{D}(X_\mathrm{D}(t))$ whose dimension is $\mathrm{dimz}(n) = n(n+1)/2$, is composed of products $X_{\mathrm{D}i}(t)X_{\mathrm{D}j}(t)$. In the following, subscript D is omitted to simplify notations. Hence RM is written:

$$\begin{cases} \dot{X}(t) = FX(t) + GU(t) + \Omega Z(X(t)) \\ \widehat{Y}(t) = HX(t) \end{cases} \tag{26a,b}$$

with

$$Z(X(t)) = \begin{bmatrix} X_1^2(t)\ X_1(t)X_2(t)\ X_1(t)X_3(t)\ \cdots\ X_1(t)X_n(t) \\ X_2^2(t)\ X_2(t)X_3(t)\ \cdots\ X_2(t)X_n(t)\ \cdots\ \cdots \\ X_{n-1}^2(t)\ X_{n-1}(t)X_n(t)\ X_n^2(t) \end{bmatrix}^\mathrm{T} \tag{27}$$

In many reduction methods, states are computed and a selection of some dominant states is made using different techniques: see for example, the case of modal methods where the selection of eigenmodes can be made by simple truncation [1] (the eigenvalues corresponding to the large time constants are kept) or by considering energy criteria [2,3] (modes carrying the maximum of energy according to the chosen criterion are kept).

In the present paper, modes are not computed by solving an eigenvalue problem and hence modes are not selected as in modal methods [1–4], for example. Components of matrices, including the n dominant eigenvalues (matrix F) and eigenvectors (matrix H), are identified through a minimization procedure explained in the following Section 4 along with the choice of the order n.

The following transformation is applied:

$$X'(t) = \mathrm{diag}(G_1, \ldots, G_n)X(t) \tag{28}$$

where $\mathrm{diag}(G_1, \ldots, G_n)$ is the diagonal matrix whose components are the G_i, $i = 1, \ldots, n$. We get:

$$\begin{cases} \dot{X}'(t) = FX'(t) + \underline{\mathbf{1}}U(t) + \Omega' Z(X'(t)) \\ \widehat{Y}(t) = H'X'(t) \end{cases} \tag{29a,b}$$

where $\underline{\mathbf{1}}$ is the vector whose dimension is n and all components are equal to 1, $\Omega' = [\mathrm{diag}(G_1, \ldots, G_n)]^{-1}\Omega\,\mathrm{diag}(Z(G))$ with $Z(G)$ (dimension $\mathrm{dimz}(n)$) gathering products G_iG_j, and $H' = H\,\mathrm{diag}(G_1, \ldots, G_n)$.

By omitting prime symbol to lighten notations, the final form of the Reduced Model is obtained:

$$\begin{cases} \dot{X}(t) = FX(t) + \underline{\mathbf{1}}U(t) + \Omega Z(X(t)) \\ \widehat{Y}(t) = HX(t) \end{cases} \tag{30a,b}$$

where:

- $X(t)$ (dim. n) is a low dimensional state vector ($n \ll N$), $\dot{X}(t)$ its derivative with respect to time. States $X_i(t)$ are coupled through nonlinearities.
- $Z(X(t))$ (dim. $\mathrm{dimz}(n) = n(n+1)/2$) is the vector of nonlinearities gathering products $X_i(t)X_j(t)$.
- $U(t)$ is the thermal input.
- $\widehat{Y}(t)$ (dim. q) is the approached output vector (such as $\widehat{Y}(t) \approx Y(t)$).
- F (dim. n, n) is a diagonal matrix containing n characteristic "eigenvalues" to be *identified*.
- Ω (dim. $n, \mathrm{dimz}(n)$) is the matrix quantifying the contribution of components of nonlinear vector $Z(X(t))$ in each one of the n coupled equations. Components of Ω have to be *identified*.
- H (dim. q, n) is a reduced output matrix which has to be *identified*.

4. Reduced Model identification algorithm

The identification of components of matrices F, Ω and H is realised through the minimization of a squared residues functional J_{red} built with the discrepancy between responses of the system (in this study, outputs of DM defined by Eq. (5a,b) on the one hand and outputs of RM defined by Eq. (30a,b) on the other hand, when a specific input signal is applied:

$$J_{\mathrm{red}}(n, F, \Omega, H) = \sum_{i=1}^{q} \sum_{j=0}^{nt} (Y_i(t_j) - \widehat{Y}_i(t_j))^2 \tag{31}$$

where nt is the number of time steps in the simulations and the t_j are the discretization times.

RM is identified from simulations made with a DM, meaning that RM is built to fit in with DM's responses. There are two consequences:

1. First, RM will be as good as the DM used for the computation of data. In fact, from a coarse DM, only a coarse RM can be obtained, and from an accurate DM, you can get an accurate RM.
2. Second, the fact that RM fits in well with DM (i.e. the value of quadratic criterion J_{red}) depends on the order n of RM and on the precision wished by the user. Starting with $n = 1$, n is incremented and a RM is identified for each value of n until the gap between two successive criterions is very low or until the mean quadratic error $\boldsymbol{\sigma}_{\mathbf{Y}}^{\mathrm{id}} = \sqrt{J_{\mathrm{red}}/q(nt+1))}$ between RM and DM's responses is of the order of magnitude of the precision wanted by the user (cf. following Section 5 for an application).

Remarks:

1. If the RM is built from measured data instead of DM's simulations, iterations of the minimization algorithm should be stopped when $\boldsymbol{\sigma}_{\mathbf{Y}}^{\mathrm{id}}$ reaches the order of magnitude of measurement errors of data used for identification.
2. If the RM is going to be used for solving inverse problems (cf. part II of the paper [18]), it can fit in with DM with the order of magnitude of the measurement errors of data used for inversion.
3. RM's order n depends more on the dynamics of the physical points chosen as outputs than on the number of chosen points. If the dynamics of the chosen points are quite different, RM's order is larger than if the points have similar dynamics.
4. RM is identified from temperature evolutions of a number q of chosen physical points that are of interest for the user, hence it is built to simulate the dynamics of these q points specifically. In fact, the first dimension of identified output matrix H is always equal to the number q of chosen points, meaning that RM can simulate only these q points. If other points are chosen, another RM should be identified from corresponding temperature data. Of course, it is possible to build a RM for any $q \leqslant N$ and then a priori for the N points of the original DM, but it could be long in terms of computational cost as a large amount of data would be used. Usually, in practical applications, only a few points are used.

In contrast to linear systems for which a RM identified from responses to any known input signal will be a priori valid for any other input signal, nonlinear systems basically react in a different way according to the excitation level. Consequently, a RM identified from responses to

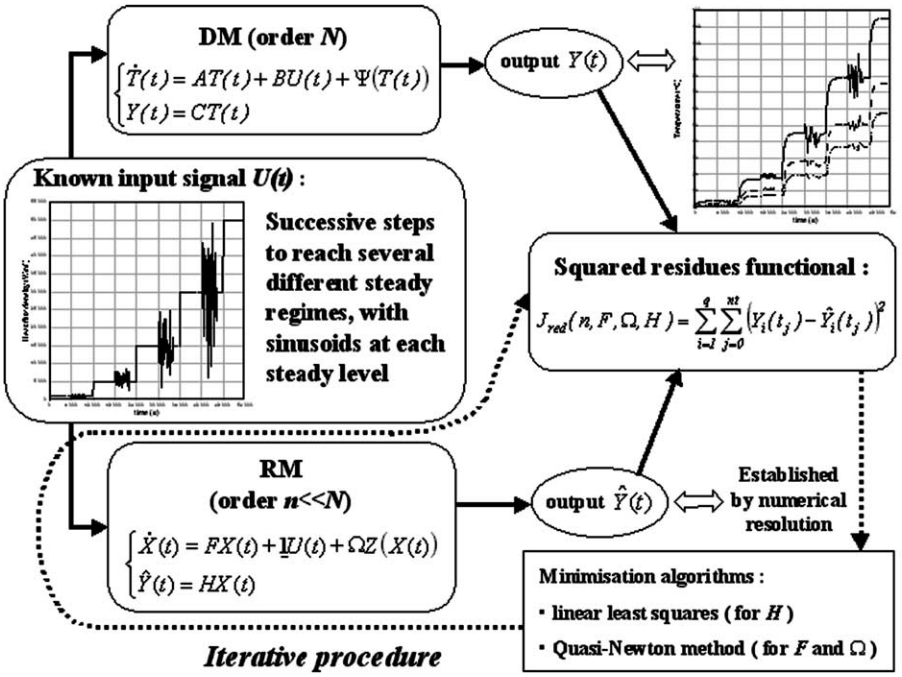

Fig. 2. RM identification algorithm.

a known input signal $U_1(t)$ will not necessary adequately reproduce dynamics of the system when a different input signal $U_2(t)$ is applied. The signal used to generate data for the RM identification must allow the system to react in large ranges of temperature levels and frequencies. That is why we propose to use signals composed of successive steps to reach several different steady regimes, with random values around each steady level in order to exhibit some frequencies in the system dynamics.

The method for minimizing the squared residues functional J_{red} uses optimization techniques. The vector $\widehat{Y}(t)$ being nonlinear with respect to matrices F and Ω, an iterative method of the *quasi-Newton* class is used for the identification of components of F and Ω. $\widehat{Y}(t)$ being linear with respect to matrix H, components of H are obtained using *linear least squares at each iteration*. The identification algorithm requires numerous resolutions of nonlinear first order differential equations (30a,b). The low number n of equations involved, together with the diagonal aspect of matrix F, ensure very small computing time for each of these resolutions. The identification procedure scheme for a given order n is given in Fig. 2.

5. An example of application

We propose a numerical example to illustrate the identification method. A 3D system, a cube ($0.1\,\text{m} \times 0.1\,\text{m} \times 0.1\,\text{m}$) shown in Fig. 3, is considered. It is composed of a material whose thermal conductivity depends linearly on temperature according to the following law:

$$\lambda(T) = 16(1 + 0.01(T - 20)) \tag{32}$$

where the local temperature T is expressed in °C.

The transient nonlinear energy equation is written:

$$\rho C_p \frac{\partial T}{\partial t} = \operatorname{div}(\lambda(T)\,\vec{\operatorname{grad}}\,T) \tag{33}$$

where $T = T(x, y, z, t)$ and $\rho C_p = 4.029 \times 10^6\ \text{J m}^{-3}\,^{\circ}\text{C}^{-1}$.

Associated boundary conditions are written:

$$-\lambda(T)\frac{\partial T}{\partial x} = U(t) \quad \text{at } x = 0 \tag{34a}$$

$$-\lambda(T)\frac{\partial T}{\partial x} = h(T - T_a) \quad \text{at } x = 0.1 \tag{34b}$$

$$T = 0 \quad \text{at } y = 0 \tag{34c}$$

$$\lambda(T)\frac{\partial T}{\partial y} = 0 \quad \text{at } y = 0.1 \tag{34d}$$

$$\lambda(T)\frac{\partial T}{\partial z} = h(T - T_a) \quad \text{at } z = 0 \tag{34e}$$

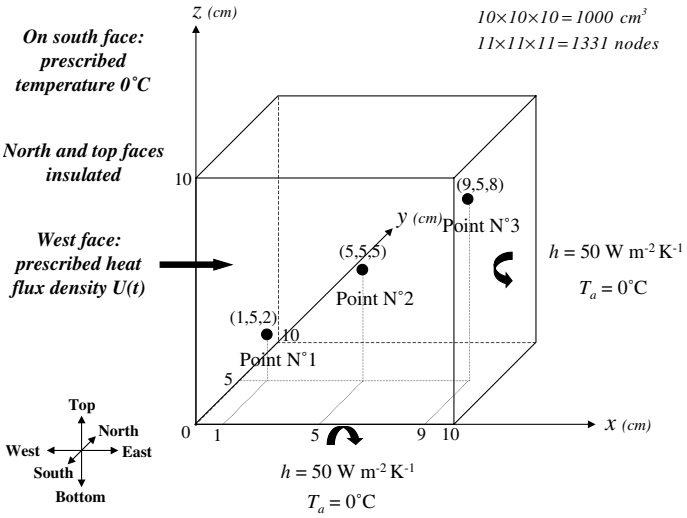

Fig. 3. System description.

$$\lambda(T)\frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0.1 \tag{34f}$$

where $T_a = 0\,^{\circ}\mathrm{C}$ is the ambient temperature surrounding east and bottom faces, and $h = 50\,\mathrm{W\,m^{-2}\,^{\circ}C^{-1}}$ is a convective exchange coefficient.

A possible initial condition is given by the resolution of Eq. (33) in steady state when boundary conditions ((34a)–(f)) are applied with $U(t = 0)$.

The domain is discretised using the Finite Volumes Method, with 11 nodes in each direction. Consequently,

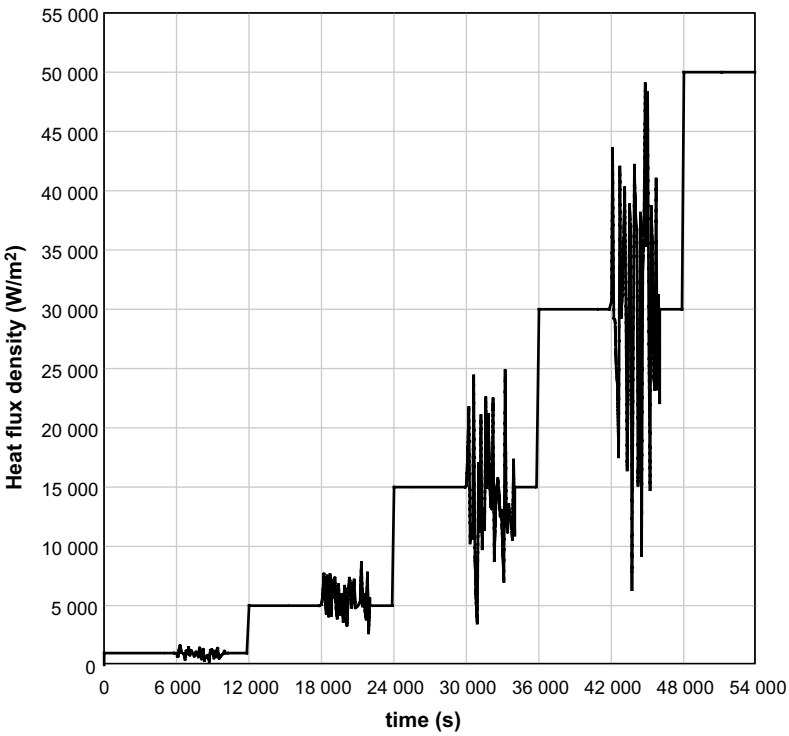

Fig. 4. Input signal $U(t)$ used for RM identification.

a DM of order $N = 1331$, taking the form of Eq. (5a,b), is built.

In order to illustrate the method, let us consider three points inside the domain (cf. Fig. 3), the first one located near the bottom face and the west face submitted to the applied heat flux density $U(t)$, the second one located at the centre of the cube and the third one near the top face and the east face opposed to the heated boundary. We wish to build a RM describing the thermal behaviour of these three points. In fact, RM's outputs are the points that are interesting for the user for its particular application. The reader should then note that in Part II of the paper [18], the RM identified in the present Part I will be used for solving an inverse problem with three sensors that are located at the points for which RM has been built.

In Fig. 4 is shown the signal $U(t)$ used for the RM identification. nt = 10800 time steps of 5 s are used, for a total of 32403 data for the temperature evolutions of the three points. Fig. 5 shows the temperature response at point No. 1 obtained with the DM of order $N = 1331$. In Table 1 are summarized the results of the identification of RMs of orders $n = 1$–5, that is, the minimization of the quadratic criterion J_{red} (Eq. (31)) for each of these values of n. For $n = 1$–3, the mean quadratic error $\boldsymbol{\sigma}_{\mathbf{Y}}^{\text{id}} = \sqrt{J_{\text{red}}/(q(nt+1))}$ characterising the RM identification decreases rapidly from 1.4 to 0.035 °C. For $n = 4$, the gain in precision is still substantial with $\boldsymbol{\sigma}_{\mathbf{Y}}^{\text{id}} = 0.02$ °C. Increasing the order to $n = 5$ gives $\boldsymbol{\sigma}_{\mathbf{Y}}^{\text{id}} = 0.015$ °C: the identification criterion is slightly better but the improvement is not significant. That's why the RM of order $n = 4$ will be used in the following. In Fig. 5 is also shown the residual between DM's and RM's response at point No. 1 (multiplied by 100 to ease the reading). Of course, some error peaks corresponding to steps of the input signal $U(t)$ can be observed, but the identified RM reproduces adequately the DM thermal behaviour for the point No. 1 when the system is submitted to the specific input shown in Fig. 4. Similar observations can be made with the two other points.

In order to validate the identified RM, it is necessary to test it with input signals $U(t)$ very different of the signal used for the model identification. Two test examples are presented. Fig. 6 shows the first signal $U(t)$ which

Table 1
Summary of RMs identification results: mean quadratic error $\boldsymbol{\sigma}_{\mathbf{Y}}^{\text{id}}$ versus RM's order n

RM's order n	1	2	3	4	5
$\boldsymbol{\sigma}_{\mathbf{Y}}^{\text{id}}$ (°C)	1.396	0.191	0.035	0.020	0.015

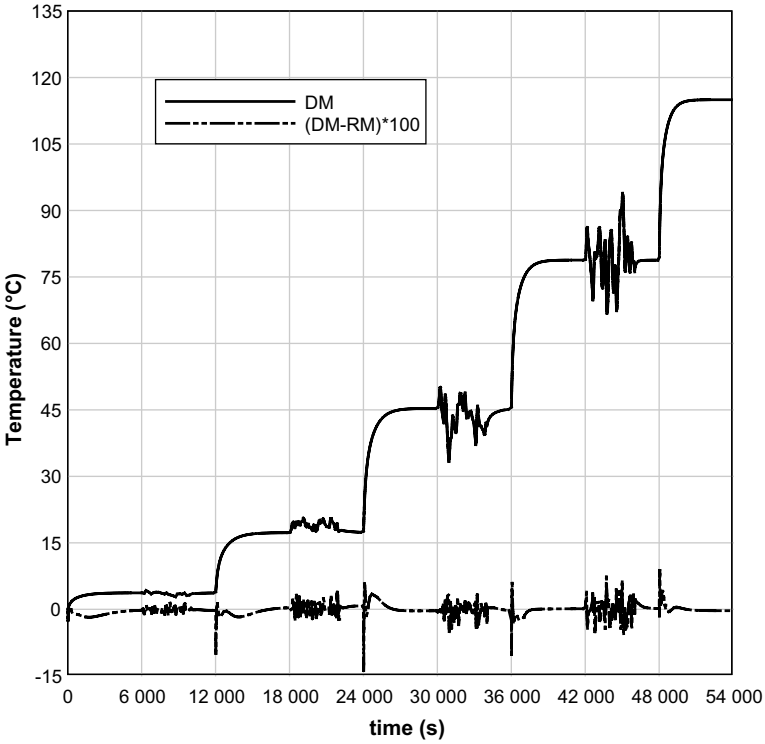

Fig. 5. Temperature evolution at point No. 1, obtained with DM as well as RM of order 4 when the input signal used for RM identification (Fig. 4) is applied. Residual between responses (×100).

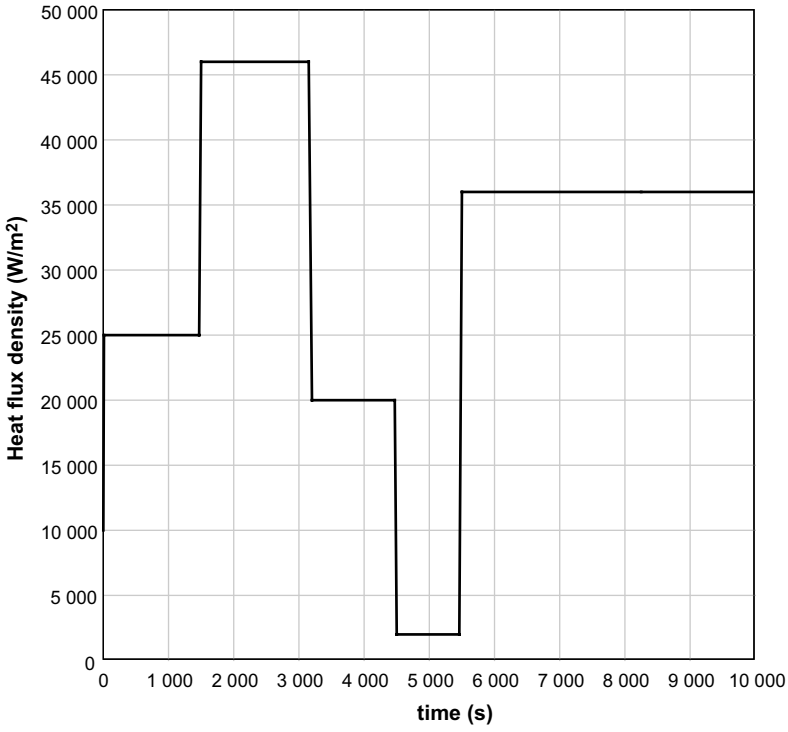

Fig. 6. Test function No. 1 for $U(t)$.

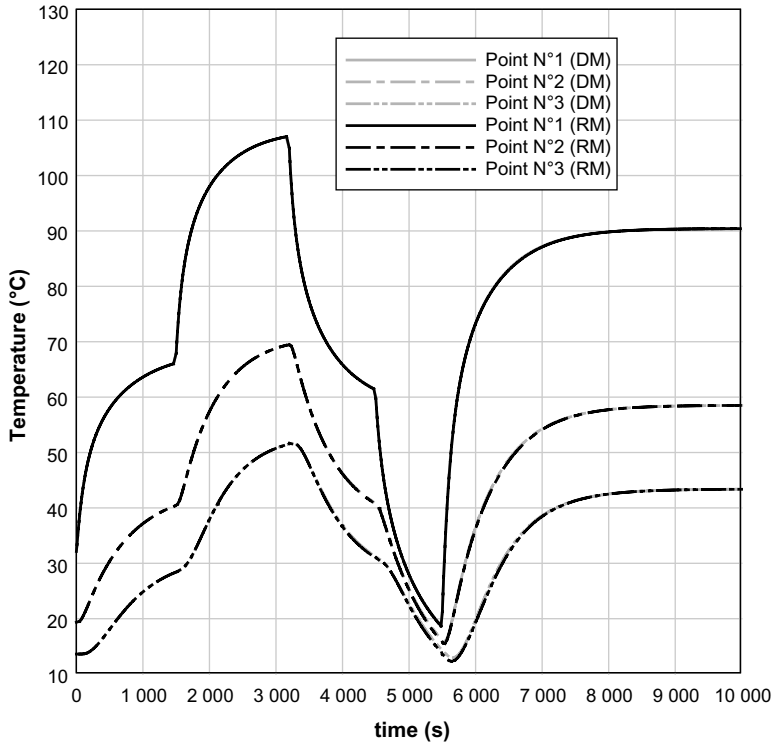

Fig. 7. Temperature evolutions obtained with both DM and RM when test function No. 1 is applied.

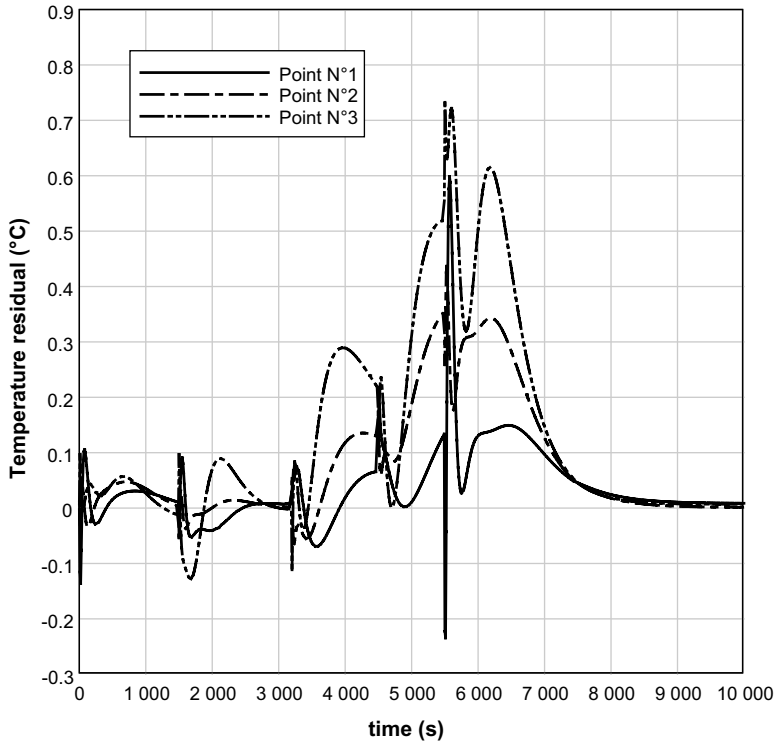

Fig. 8. Discrepancies between DM and RM responses when test function No. 1 is applied.

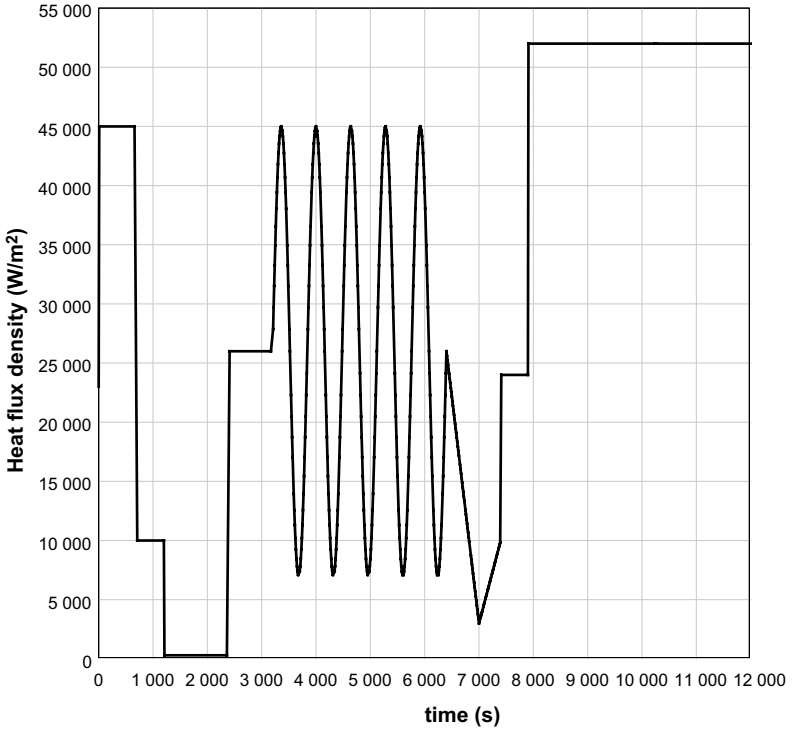

Fig. 9. Test function No. 2 for $U(t)$.

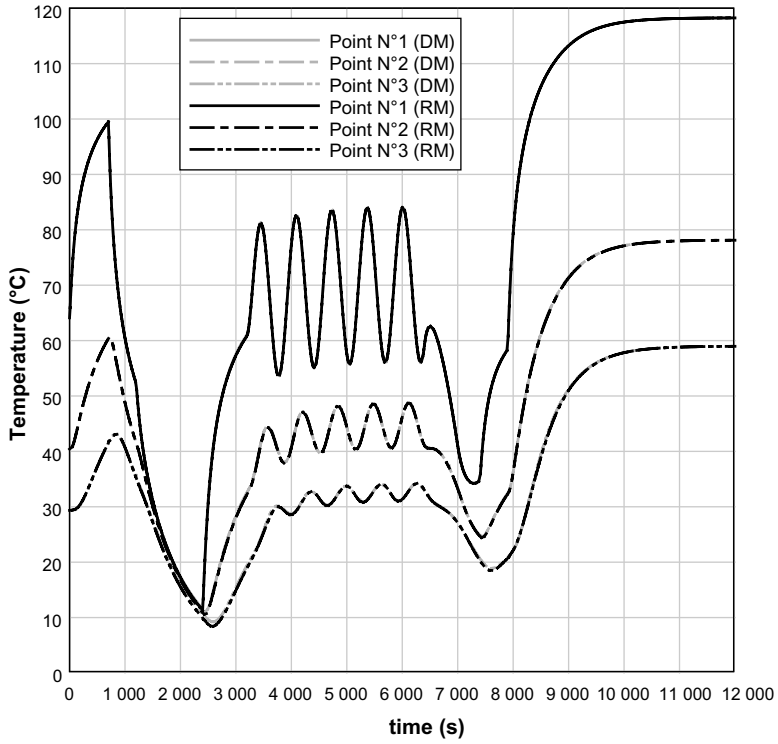

Fig. 10. Temperature evolutions obtained with both DM and RM when test function No. 2 is applied.

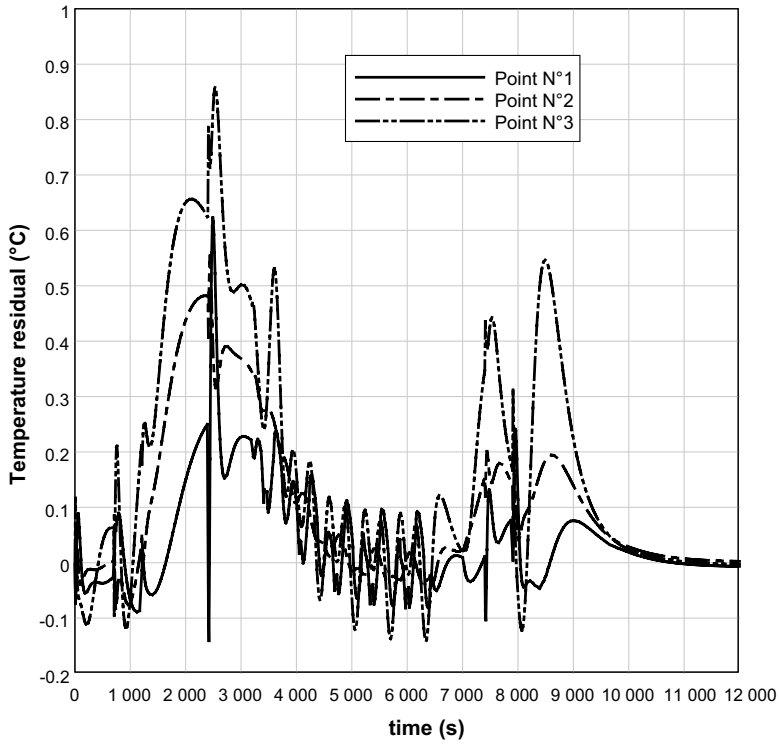

Fig. 11. Discrepancies between DM and RM responses when test function No. 2 is applied.

contains only increasing and decreasing steps. Fig. 7 shows that temperature responses computed with RM are quasi perfectly superposed to those obtained using DM. Fig. 8 shows the corresponding residual between DM's and RM's responses.

The second test function is shown in Fig. 9: the signal $U(t)$ is more complex, including a sinusoid (whose amplitude and period are different from those of the signal used for the identification) and ramps. Again, responses obtained using RM are in excellent agreement with those computed with DM (cf. Fig. 10). Corresponding residual is given in Fig. 11.

Computing time has been divided by a factor greater than 1000:0.15 s CPU with RM versus 163 s CPU with DM.

Finally, one should note the importance of nonlinearities in the presented example. In fact, if the linear RM obtained by zeroing the nonlinear term $\Omega Z\ X(t)$) in Eq. (30a) is used, resulting temperature evolutions are far from those shown on Figs. 7 and 9, with discrepancies up to 80 °C in the case of test function No. 2.

6. Conclusion

A new method for identifying low order models of nonlinear diffusive thermal systems has been proposed. The structure of the Reduced Model equations is obtained by manipulating equations of a Detailed Model under State Space Representation when a discretization of the domain is made using a classical method (finite elements, finite volumes, finite differences,...). The RM links some chosen temperature outputs in the domain to a single thermal input. Matrices of RM equations are identified through the minimization of a squared residues functional built with the discrepancy between responses of the system (in this study, outputs of a DM) in one hand and outputs of RM on the other hand, when a specific input signal is applied. The identification algorithm uses optimization techniques.

Thanks to the adopted state space representation, the method is independent of geometry and can be applied to multidimensional systems. RMs work in both steady and transient regimes.

Although the method has been developed for systems with thermal conductivity linearly dependent of temperature, it has been underlined that extensions could be written in the more general hypothesis of thermal conductivity depending on temperature according to a polynomial law and also to the case of thermal capacity varying with temperature.

A numerical example of a simple tridimensional system is presented to illustrate the method, but the method can be applied to any complex geometry if a DM can provide the simulations needed to identify the RM.

The proposed approach for identifying nonlinear thermal systems seems promising: substantial reduction of computing time, use of a RM for the resolution of inverse problems (first tests have already been successfully performed [18]), identification of a RM from experimental data substituted to simulated numerical data.

The extension of the reduction method to problems involving more than one independent time-varying unknown is at this time under development, as well as an extension to Navier–Stokes equations for the identification of coupled nonlinear Reduced Models describing a flow at some specific locations.

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